CONNECTIONS BETWEEN QUALITATIVE AND QUANTITATIVE THINKING ABOUT PROPORTION: THE CASE OF PAULINA

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Abstract

The case study presented in this report was part of assessing a teaching proposal on ratio and proportion. A group of sixth-grade students of elementary education in México participated in the implementation of the proposal. The girl of the case study was representative of those students in the group who had a lot of recourse to handling algorithms mechanically and whose elaborations made no sense at all, according to their answers to an initial questionnaire. A didactical program, developed in a problem-solving context for the research study, helped the girl widen her qualitative thinking and strengthen her quantitative thinking about proportion. Analyses of data collected from an initial questionnaire, the teaching process, a final questionnaire, and three interviews evidenced that enriching this girl's qualitative thinking about proportion allowed her widen quantitative relations and improve her handling of algorithms by providing the setting for meaningful applications.

Some theoretical antecedents of the investigation

Piaget and Inhelder (1978) pointed out, as a result of their experimental researches in education, that children acquire qualitative identity sooner than quantitative conservation. Thus, these authors made a distinction between qualitative comparisons and true quantification. According to Piaget and Inhelder (1972), the acquisition of the notion of proportion always starts in a qualitative and logical form before it becomes quantitatively structured. Piaget defined what is qualitative by using categories or classes of words. Our own interpretation of what is qualitative refers to what is based on linguistic recognitions by creating comparison categories such as big or small. Our interpretation is that what is qualitative consists of intuitive and empirical aspects as well, which are provided by our senses.

Piaget (1978) pointed out that the idea of order emerges during the transition from the qualitative to the quantitative realm, although the idea of quantity is not yet present. Piaget called these situations *intensive quantifications*. For us, this is what makes the transition from qualitative to quantitative thinking stand out.

On their part, Van den Brink and Streefland (1979) agreed with Piaget as to their research findings that qualitative aspects of thinking occur sooner than quantitative ones. However, Streefland usually had recourse to these findings in teaching contexts. In our approach for the designing of the didactical program as well as in the development of interviews for the case study of educational research we present in this report, we used that contribution by Streefland.

Research findings reported by Streefland (1984; 1985) emphasized that the early teaching of ratio and proportion topics must depart from qualitative levels of recognizing them. For that purpose, Streefland made use of didactical resources, which strengthen the development of perceptual patterns for supporting the corresponding processes of quantification. Streefland stated that qualitative reasoning evolves as the thinking of the child advances and he or she is capable of incorporating

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more elements for an analysis, which will allow him or her to consider different factors simultaneously.

Thus, since Piaget and Streefland took into account qualitative and quantitative thinking about proportion exhibited by their subjects under research, the rationale for our case study was strongly based on Piaget's and Streefland's findings. We based the didactical approach developed for our research on Streefland's realistic mathematics approach.

Hart (1988) and her collaborators had reported results of their research studies on proportional thinking as well. They found out that most students who participated as subjects in their researches considered that it was difficult to solve mathematics problems that involved proportion. However, Hart and her team analyzed collected data and evidenced that younger students as well as pupils in secondary school with less success had a certain sense of "what is seen right" or of "what seems to be a distortion." Hart designated the latter as a regulation from "common sense," which we recognized as intimately involved in "qualitative thinking." Moreover, Hart pointed out that the most advanced level of proportional thinking occurred in those subjects who had already constructed certain concepts.

We based the didactical context of our research on realistic mathematics education referred to by Streefland (1993). Realistic mathematics education has become a theory since reality is, in first instance, a source of information and the context for the application of teaching models, schemata, and notations—school productions that have an influence in social practice. This theory favors the development of research and practice of the teaching and learning of mathematics. Analogously, according to this realistic theory it is essential to link students' learning periods by resorting to the "strategy of change in perspective," which is characterized by the exchange of part of the information in the problem-situation being approached. Consequently, the possibilities for the reconstruction and production of problems become explicitly recognized by students, without losing their multifaceted conceptual richness.

Research problem

The case study we present in this report was part of a research study carried out for a doctoral dissertation (Ruiz Ledesma, 2002). Previously, other aspects and activities of that research have been presented and reported in various communications. The case study of our research is about a girl, Paulina, who solved ratio and proportion problems by having recourse to algorithms which made no sense and had no meaning at all.[1]

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1 According to Benveniste (1971), meaning is a "dictionary entry" and "a universal semantic category"; and sense is a semantic content, which is associated to particular constructions of language, it does not shape universal categories and usually keeps a close relation to specific modes of articulating them. Moreover, it is proper to emphasize that there is not a chronological sequence, or of precedence, in the development of sense and meaning. They are different semantic components, which complement each other.

We designed a teaching proposal embedded in this situation, with the aim of strengthening her establishing of solid connections between qualitative and quantitative thinking about proportion, so that she could improve her handling of algorithms by situating them into meaningful applications. The following question guided our research about Paulina's case.

Research question

Does the extensive handling of qualitative aspects of ratio and proportion allow the student to widen quantitative relationships of these concepts as well as to improve the handling of her algorithms?

Hypothesis

Enriching Paulina's qualitative thinking—by using integrated verbal categories, recognizing the compensations posed between these categories, and involving the corresponding empirical and perceptual data—favors the significance processes she has developed by using algorithms for solving ratio and proportion problems.

Methodology

The research process of the case study of Paulina included integrating results from analyses of data collected from (a) her answers to an initial questionnaire, (b) a teaching program designed under a constructivist-didactical approach, (c) a final questionnaire, and (d) interviews of "didactical nature." The interviews with Paulina were based on results presented by Valdemoros (1998). The research instruments were tested in a pilot study of a one-year school cycle and definitively implemented during a ten-month period of fieldwork. In this case-study report we present relevant examples of the use of the research instruments.

The initial questionnaire was applied to collect evidence of qualitative thinking about proportion. The tasks included in this questionnaire did not involve the use of quantities for their solution: it comprised comparison activities that allowed the student recognize similarity relationships between figures.

Figueras, Filloy, and Valdemoros (1987) defined *model* as a collection of teaching strategies which include meanings—of both technical and common languages—, didactical treatments, specific modes of representation, and their interrelations. In the didactical program, according to that definition, we designed several situations associated to "teaching models" so that Paulina could link her qualitative and quantitative thinking processes on proportion. We worked with those models at different stages of the research experiment, similarly to what Streefland (1993) pointed out in his realistic theory as to the "change strategy in perspective:" We created a model and tried to get the best out of it in the light of an idea, so that we could retake it and use it for another idea.

Twenty-nine students of sixth-grade of elementary education in México, who were eleven years old, solved the initial questionnaire. We chose Paulina for a case study because she was representative of those students who, in the initial questionnaire, had a lot of recourse to handling algorithms that made no sense and who simultaneously exhibited few elaborations in the qualitative context. Throughout the development of the teaching experience, Paulina exhibited enrichment of her qualitative thinking and, in spite of making a lot of progress in the numerical context, she did not abandon the qualitative context of proportionality. She achieved a close harmony of both contexts.

Analysis of Paulina's progress by comparing her answers in the initial and in the final questionnaires

The initial and final questionnaires were integrated by the same tasks, although their application had a different aim. The first questionnaire was applied for exploratory purposes, whereas the second one focused on evaluating the implementation of the teaching program. Eight months elapsed between the applications of both questionnaires: Thus, there was no influence of the first questionnaire on the students' answers to the second one.

In the initial questionnaire, Paulina exhibited a preference for using algorithms mechanically and very little work in the qualitative context. We observed that she almost did not use her common sense or visualization. From the thirteen tasks posed in the initial questionnaire, she solved nine of them correctly.

The first two tasks in the questionnaire were designed so that Paulina could give justifications of her answers by strongly resorting to qualitative appreciations and not taking into account explicit quantities associated to the given relationships of proportionality. We employed squared paper in the next three tasks of the questionnaire to favor a transition toward quantification. The remaining tasks in the questionnaire involved quantified situations of ratio and proportion. In these last tasks, we provided Paulina with certain numerical values and asked her for new values. In some of these tasks we used a table of numerical values as a mode of representation for the recognition of external and internal ratios. Now we present an analysis of two tasks Paulina answered incorrectly: task 1 and task 4.

In task 1, the drawing of a house was presented and the student was required to select the correct reduced sketching of the original drawing (see figure 1). Paulina selected a sketching that did not correspond to the original drawing and she argued that her choice resembled best the original drawing of a house. However, in the final questionnaire Paulina based her choice of the reduced drawing by having recourse to her intuition first and then by measuring each part of this drawing to obtain the ratios with corresponding magnitudes of the original drawing, although in her new explanation she mentioned again that "house C looks like Antonio's house" and added that "it is similar, that is, proportional" (see figures 1 and 2). Thus, we observed that, from the initial questionnaire to the final one, the expression "looks like" underwent a change of meaning for Paulina: she exhibited an understanding of the term "proportion" as the relationship of equivalence between two ratios (but she did not abandon her common sense, which was exploited throughout the teaching program). We can ascertain this based on other collected evidence: for instance, Paulina did not solve correctly task 4 in the initial questionnaire, but she did in the final one. It is important to make the explanations she elaborated stand out in this research study; they are included in figures 4 and 5.

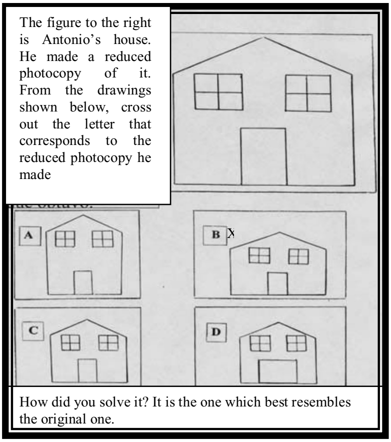

Figure 1. Task 1 of the initial questionnaire solved by Paulina.

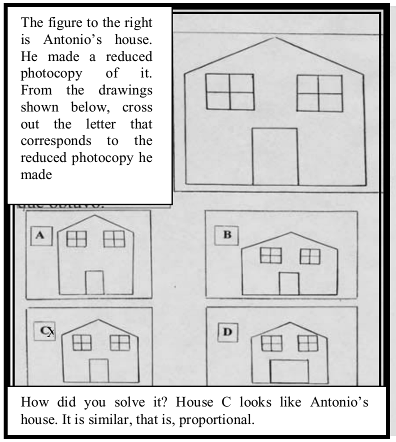

Figure 2. Task 1 of the final questionnaire solved by Paulina.

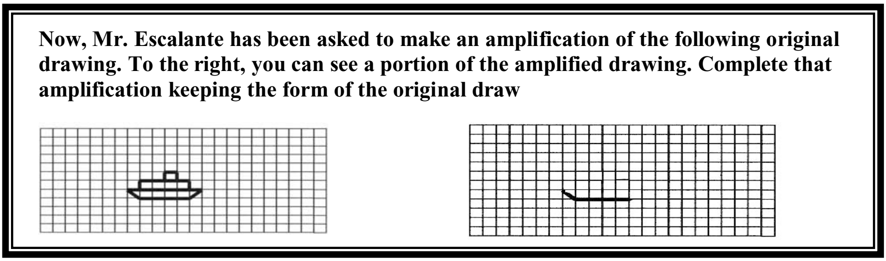

Figure 3. Task 4 in the initial and final questionnaires.

As shown in figure 4, Paulina completed the drawing but she did not notice that she had amplified it twice and not thrice. As seen from figure 5, Paulina showed the establishing of equivalence between two ratios that were obtained from comparing two corresponding magnitudes from the middle portion of the ship.

Analysis of Paulina's progress during the development of the teaching program

The solution of different tasks employed during the development of the teaching program, such as comparison activities, involved using quantities. These activities allowed Paulina recognize—by using very intuitive terms such as reduction and amplification—similarity relationships between figures and she could enrich her qualitative thinking. We worked with those notions by referring to concrete situations of the type of the experience of reproducing a drawing to scale and of the idea of using a photocopier.

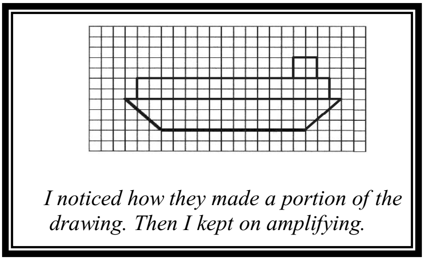

Figure 4. Task 4 solved by Paulina in the initial questionnaire.

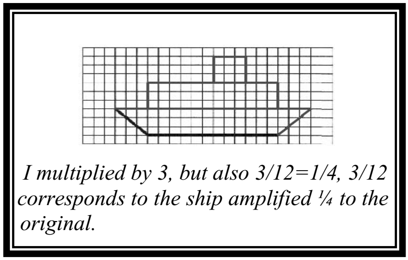

Figure 5. Task 4 solved by Paulina in the final questionnaire

The solution of different tasks employed during the development of the teaching program, such as comparison activities, involved using quantities. These activities allowed Paulina recognize—by using very intuitive terms such as reduction and amplification—similarity relationships between figures and she could enrich her qualitative thinking. We worked with those notions by referring to concrete situations of the type of the experience of reproducing a drawing to scale and of the idea of using a photocopier.

During the transition from qualitative to quantitative thinking, Paulina produced an ordering when comparing: she used the phrases "bigger than and smaller than". This finding agrees with what Piaget (1978) pointed out. Later on, Paulina took measures to make comparisons. First, she compared different objects by placing one figure over another and then by using a measure instrument. In terms stated by Freudenthal (1983), the resources exhibited by Paulina at this development stage of her thinking are called "comparers." After that, Paulina established relationships between magnitudes. She worked with natural numbers and employed fractions as well. Thus, at a very elementary level, she introduced herself to the field of rational numbers. The girl of this case study could designate a ratio as a relation between two magnitudes and a proportion as an equivalence relation between two ratios. This designation agrees with definitions given by Hart (1988).

When the working sessions ended, Paulina showed she had achieved a close relationship between her qualitative and quantitative thinking. This relationship implied the sense she made of her work in the numerical context, which was not revealed at the beginning of her work. Eventually, when the teaching experience ended and the final questionnaire was applied, Paulina's meanings and quantification processes had been enriched. Now she could use a technical language in the designation context. She achieved a generalization stage in which new situations related to ratio and proportion were favored.

Analysis of Paulina's progress during the interviews

Paulina was interviewed in three different occasions, once a week, after the teaching program ended and the final questionnaire had been applied. The main purpose of the interviews was to asses the teaching program. The interviews consisted of asking Paulina to solve new tasks which aims were similar to those of the didactical program and of the questionnaires. Additionally, the development of the interviews gave feedback to Paulina.

With the first tasks we posed Paulina during the interviews, through her solution processes we could observe how she kept qualitative aspects to the light of having worked quantitative aspects, and how important it was for her to use visual images as well as her perception ability. Through the next tasks in the interviews, we also investigated how she handled numerical tables to recognize ratios and express these as fractions. During the interviews she exhibited her use of internal and external ratios, her transition from one symbolic system to another, and her posing of a situation where the use of proportions would be necessary to solve it.

This first interview was closely related to the Snow White and the seven dwarfs teaching model. Next, we show the development and analysis of that interview.

Paulina measured the length and the width of Snow White's wardrobe as well as the length and the width of each of the four drawings shown in the figure so that she could choose the required reduction. Once Paulina had chosen a wardrobe, she obtained the ratios between magnitudes of some of its parts and the corresponding parts of the original wardrobe. Now we show part of the interview with Paulina.

Interviewer: What did you base your choice of the dwarfs' wardrobe on?

Paulina: I took measures and found out that wardrobe B is proportional to Snow White's because all their ratios are equivalent. (Paulina pointed to what she had written, "12/8 = 6/4 =3/2.")

Interviewer: Will you please tell me how you obtained the ratios?

Paulina: By comparing measurements of Snow White's wardrobe with those of the dwarfs'. The numerator of each fraction measures certain part of Snow White's wardrobe: for instance, 12 is the length of the height, 8 is the length of the base, 3 is the length of one little window (she pointed to one of the drawings representing a decoration of the wardrobe), and 1.5 is the width of this little window. The denominators of the fractions are the measurements of the corresponding parts of the dwarfs' wardrobe. (The measurements Paulina mentioned are given in centimeters.)

Thus, Paulina established links to determine ratios based on taking measures. In another part of the same interview we could observe how she had recourse to her perception ability when she said, "Wardrobe A is too long, C is very wide, and D is very little. Although I did take measures, I noticed that those three wardrobes did not seem proportional to Snow White's."

Paulina exhibited that her handling of conceptual aspects was meaningful since she identified ratio as a relation and proportion as an equivalence relation between ratios. Moreover, we could notice that Paulina did not abandon the qualitative context, since she also used verbal categories and common sense to verify that her choice of the wardrobe was the right one. To this respect, she wrote that Snow White's wardrobe was equivalent to that of the dwarfs, and that as to their form they were equal although one was small and the other was big

Conclusions

Paulina exhibited a strong progress in relation to two important aspects:

1. The development of her qualitative thinking in relation to ratio and proportion.
2. The signification she gave to her using of algorithms.

During the processes of solving different tasks, Paulina exhibited how strong perceptual data became for her as well as how important it was for her to rely on her own experience. This is evidence about her achievements in the qualitative context of proportionality. The algorithmic work allowed us to explore the tacit recognition of the operators about which Paulina was thinking. These operators were natural numbers as well as fractions. The latter were used implicitly when multiplying certain value by a number and then dividing the result by another number, or vice versa, first dividing and then multiplying. In the context of what is now considered the construction of meanings, these—together with the processes of signification—were enriched. As to their designation, Paulina could eventually use the appropriate mathematical terms. Finally, she reached the point of constructing the concepts of ratio and proportion. This achievement was evidenced by the applications she made of those concepts in different contexts as well as by using their different modes of representation.

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